Astro 501, Fundamental Astronomy Problem Set 7

Due December 7

1. An Sc disk galaxy has the following properties:

I-band major-axis disk scale length ...	$50''$
I-band central surface brightness	20.2 mag arcsec^{-2}
Disk axis ratio	0.4
Observed 21-cm line width	400 km s^{-1}

Assume that the I-band Tully-Fisher relation relates the absolute I-band magnitude of a galaxy to the velocity width of its 21-cm line emission by

$$M_I = -8.72(\log W - 2.50) - 20.94$$

If there is no foreground reddening, what is the distance to the galaxy? (Hint: before you go ahead with the calculation, think carefully about the best way to integrate the luminosity distribution of an intrinsically circular, but observationally inclined galaxy.)

The inclination of an intrinsically flat galaxy is given by $\cos i = b/a$, so $i = 66.4°$. The true line-width of the galaxy is then

$$W = W_{\text{obs}} / \sin i = 436 \text{ km s}^{-1}$$

Plugging this into the Tully-Fisher relation yields an absolute magnitude of $M_I = -22.16$.

Next, you need to integrate to get the galaxy's apparent luminosity. If the galaxy were face-on, this integral would simply be

$$\mathcal{L}_{tot} = \int_0^\infty 2\pi r I(r) dr$$

However, since the galaxy is inclined, its *observed* central surface brightness is more than its *true* central surface brightness by a factor of $\cos i$. (Due to inclination, the same amount of light is being squeezed into a smaller area.) So the total luminosity of the galaxy is

$$\mathcal{L}_{tot} = \int_0^\infty 2\pi r I(0) \cos i \cdot e^{-r/r_d} dr = -2\pi I(0) \cos i \cdot (r + r_d) e^{-r/r_d} \Big|_0^\infty = 2\pi I(0) r_d^2 \cos i$$

So the apparent magnitude of the galaxy is

$$m_I = m(0) - 2.5 \log\left(2\pi r_d^2 \cos i\right) = 20.20 - 9.50 = 10.70$$

The distance modulus is then

$$(m - M)_0 = m_I - M_I = 32.86 = 5 \log d - 5 \Longrightarrow d = 37.4 \text{ Mpc}$$

2. Consider a spherically symmetric virialized galaxy cluster that is 16 Mpc away. This cluster contains X-ray emitting gas which we can assume is in hydrostatic equilibrium. The galaxies in the cluster have a line-of-sight velocity dispersion, $\sigma = 900$ km s^{-1}.

a) In the absense of other heating or cooling mechanisms, what would be the temperature of the gas which is lost from the galaxies and is thermalized in the intracluster medium? (For simplicity, just assume the mean molecular weight is one.)

If the galaxies' velocities are turned into thermal motion, then

$$\frac{1}{2}\mu m_H \langle v^2 \rangle = \frac{3}{2} k T_{\text{gal}}$$

where μ is the mean molecular weight of the gas (a number that is close to 1). Since we only measure one component of the velocity dispersion

$$\frac{1}{2}\mu m_H 3\sigma^2 = \frac{3}{2} k T_{\text{gas}} \Longrightarrow T_{\text{gal}} = \frac{\mu m_H \sigma^2}{k} = 1 \times 10^8 \text{ K}$$

b) The X-ray gas doesn't have to be at the same temperature of the galaxies: infalling gas and supernovae may heat the gas, while radiative processes will cool it. Suppose the temperature of the gas is measured to be $\sim 3 \times 10^7$ K, and imaging of the gas shows that the x-ray emission declines with radius as $\epsilon_x \propto r^{-1.6}$. If the gas is roughly isothermal, what is the total mass contained within the central 1 degree of the cluster?

The mass of an x-ray cluster is given by

$$\mathcal{M}(r) = -\frac{kT}{\mu m_H G} \left\{ \frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r} \right\} r$$

Since the gas is roughly isothermal, the temperature derivative is zero, so

$$\mathcal{M}(r) = -\frac{kT}{\mu m_H G} \frac{d \ln \rho}{d \ln r} r$$

Also, since X-ray emission goes as the square of the density,

$$\epsilon_x \propto \rho^2 \propto r^{-1.6}, \Longrightarrow \rho \propto r^{-0.8} \Longrightarrow \frac{d \ln \rho}{d \ln r} = -0.8$$

At 16 Mpc, 1° corresponds to $r = 16$ Mpc $\times \sin(1^\circ) = 0.28$ Mpc. So, plugging in the numbers then gives $\mathcal{M} \sim 2.6 \times 10^{46}$ gm or $1.3 \times 10^{13} \mathcal{M}_\odot$.

3. These days, find extrasolar planets is very popular. So let's look for them in the halo of another galaxy! The spiral galaxy M31 is at a distance of 770 kpc and has an inclination of 77°. Like most spirals, M31 has a flat rotation curve ($v_{rot} = 250$ km s^{-1}), which suggests a spherically symmetric dark matter halo with an $1/r^2$ density law.) Let's assume this halo extends out to a distance of $R_{max} = 30$ kpc.

a) Assume that the dark halo of M31 is primarily composed of MACHOs (MAssive Compact Halo Objects), which could be a collection of black holes, dead neutron stars, brown dwarfs, or even planets. (Big Bang nucleosynthesis models suggest that this isn't the case, but let's try it out anyway.) For microlensing to work (or, at least obey the amplification equations), a source must fit entirely within the Einstein ring of the lensing object. So, what is the minimum mass that a distant M31 halo object must have to microlens a typical $1\,M_\odot$ main-sequence star in M31's disk? Could you detect an Earth-like planet? Justify your assumptions for this order-of-magnitude calculation.

The size of the Einstein ring is given by

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2}$$

If the entire star is to be lensed, $\alpha_0 > R/D_s$, where R is the stellar radius. For objects in M31's halo, $D_d \approx D_s = 770$ kpc. Now if we adopt $D_{ds} \approx 30$ kpc as a typical distance between the source and the lens, then

$$\mathcal{M} = \frac{\alpha_0^2 c^2 D_d D_s}{4GD_{ds}} \approx \frac{R^2 c^2}{4GD_{ds}}$$

which, when you plug in the numbers gives

$$\mathcal{M} = 8.86 \times 10^{-8} \left(\frac{R}{R_\odot} \right)^2 \mathcal{M}_\odot$$

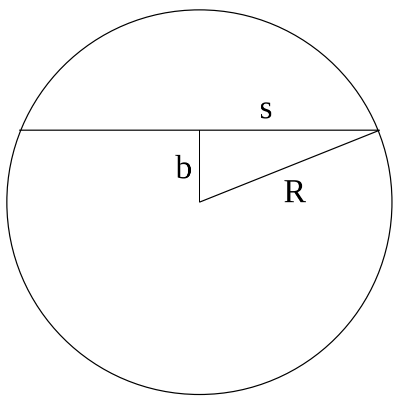
For a solar-type star with $R = 1\,R_\odot$, this is about 0.03 Earth masses, or 2.4 lunar masses.

b) How long will a typical microlensing event last if the size of the Einstein radius is greater than the size of the background star.

The length of the lensing event for an object passing through the center of an Einstein ring is given by

$$\alpha_0 = \frac{R}{D_s} = \frac{v t_{1/2}}{D_s} = \left(\frac{4G\mathcal{M}}{c^2}\frac{D_{ds}}{D_d D_s}\right)^{1/2}$$

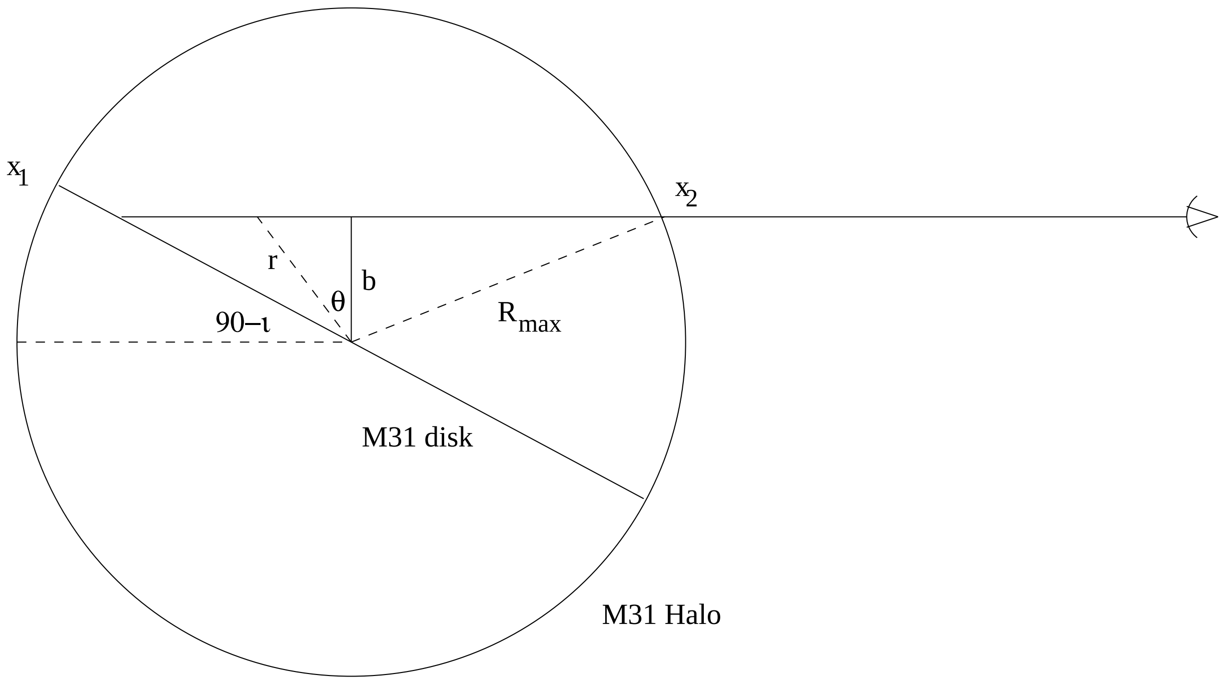
Plugging in the numbers gives $t_{1/2} = 9.3 \times 10^6$ sec, or 108 days. Since R is a radius, the total time is twice this, or 216 days.



(Note for extra credit: this is the maximum time. To find the duration of a "typical" lens event, consider that the chord an object will traverse has a length $s = \sqrt{R^2 - b^2}$, where b is the impact parameter. Also, note that the frequency of lensing events, as a function of impact parameter will go as $2\pi b db$. Thus the typical event will last

$$\frac{\int_0^R (R^2 - b^2)^{1/2} \cdot 2\pi b db}{\int_0^R 2\pi b db} = \frac{1/3(b^2 - R^2)^{3/2}\Big]_0^R}{1/2b^2\Big]_0^R} = \frac{2}{3}R$$ or two thirds as long).

c) Now consider the probability of actually observing a microlensing event caused by an M31 MACHO. Let's define M31's "optical depth" to lensing as the probability of an object falling within the Einstein ring of *some* halo lens positioned somewhere along the line of sight. Derive this value as a function of the mass of the lens and position along the minor axis of the galaxy. Are you more likely to see micro-lensing events from high-mass MACHOs or low-mass MACHOs? (Note that, since the probability of any specific lensing event is small, the total probability of observing any lensing event is simply proportional to the total area covered by all MACHOs along the line-of-sight.)



The area lensed by any one MACHO is $\sigma = \pi\alpha_0^2 D_s^2$, or, letting $D_d \approx D_s$, and $\ell = D_{ds}$

$$\sigma = \pi D_s^2 \left\{ \frac{4Gm}{c^2} \frac{\ell}{D_s^2} \right\} = \frac{4\pi Gm}{c^2}\ell$$

where m is the mass of the lensing MACHO. If the density of particles is $n(r) = n_0/r^2$ then the total halo mass (out to a distance R_{max}) is

$$\mathcal{M}_{\text{tot}} = \int_0^{R_{\max}} 4\pi r^2 \left(\frac{n_0}{r^2}\right) m\, dr = 4\pi n_0 m\, R_{\text{max}}$$

Assuming that the total area of the galaxy that is lensed at any one time is small (i.e., that the lenses don't overlap), the total area lensed along a given line-of-sight is

$$\tau = \int_{x_1}^{x_2} n(r)\sigma d\ell = \int_{x_1}^{x_2} \frac{\mathcal{M}_{\text{tot}}}{4\pi m R_{\text{max}}} \frac{1}{r^2} \frac{4\pi Gm}{c^2} \ell\, d\ell = \frac{G\mathcal{M}_{\text{tot}}}{R_{\text{rax}} c^2} \int_{x_1}^{x_2} \frac{\ell}{r^2} d\ell$$

Note that the result is independent of lens mass!

Let's now define θ as the angle between the radius vector to a point, and the normal to the line-of-sight. From the figure, we have

$$r = b \sec \theta$$
$$\ell = b \tan i + b \tan \theta$$
$$d\ell = b \sec^2 \theta d\theta$$

where i is the inclination of M31's disk, and b is the observed distance along the minor axis. Note that θ can be either positive or negative. The optical depth is now

$$\begin{aligned}
\tau &= \frac{G\mathcal{M}_{\text{tot}}}{c^2 R_{\text{max}}} \int_{x_1}^{x_2} \frac{\ell}{r^2} d\ell \\
&= \frac{G\mathcal{M}_{\text{tot}}}{c^2 R_{\text{max}}} \int_{-i}^{\arccos(b/R_{\max})} b(\tan i + \tan \theta) \frac{b \sec^2 \theta}{(b \sec \theta)^2} d\theta \\
&= \frac{G\mathcal{M}_{\text{tot}}}{c^2 R_{\text{max}})} \left\{ \int_{-i}^{\arccos(b/R_{\max}} d\theta + \int_{-i}^{\arccos(b/R_{\max})} \tan \theta d\theta \right\}
\end{aligned}$$

If we let $\theta_m = \arccos(b/R_{\text{max}})$, then

$$\tau = \frac{G\mathcal{M}_{\text{tot}}}{c^2 R_{\text{max}})} \left\{ (\theta_m + i) + \log \left(\frac{R_{\text{max}} \cos i}{b} \right) \right\}$$

(One more item to note: in general, the mass and size of any system is related to its rotational velocity by

$$\mathcal{M} = \alpha \frac{v^2 r}{G}$$

where α is of the order of unity. If you plug this in, then

$$\tau = \frac{v_{\text{rot}}}{c^2} \left\{ (\theta_m + i) + \log \left(\frac{R_{\text{max}} \cos i}{b} \right) \right\}$$

Written this way, the total mass of the halo drops out, and R_{max} only enters in the log.)